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CONTACT PROBLEM IN THE PRESENCE OF WEAR FOR A PISTON RING INSERTED INTO A CYLINDER *

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A problem of flexure of a piston ring in contact with the inner surface of a rigid cylinder and moving in the direction of its generatrix, is solved. The wear of the surfaces of the piston ring and the cylinder, its intensity, proportional to the pressure, is taken into account. Exact expressions are obtained for the ring flexure and for the contact pressure at any instant of time.

The solution obtained can be used to calculate the wear of piston rings. One of the main reasons for the loss of contact between the ring and the cylinder, is the redistribution, over a period of time, of the pressure at the circumference, in such a manner that at certain parts a contact can only be guaranteed when the piston ring groove is under a pressure of gas which presses the ring to the cylinder /1/. The pressure at the ring periphery depends, at any instant of time, on the mean value of the gas pressure, on the initial pressure diagram of the new ring, on it elastic and geometrical characteristics, and on the coefficient of wear resistance.

We assume that the rate of wear of the ring and the cylinder surfaces at any point $\partial u_*(\theta, t)/\partial t$ is proportional to the pressure $p(\theta, t)$ existing between the ring and the cylinder

$$\frac{\partial u_*(\theta, t)}{\partial t} = kp(\theta, t), \quad k > 0$$

where k is a proportionality coefficient determined by experiment. Such a relationship holds, for example, in the case of abrasive wear /2/.

Let us consider the problem of flexure of a circular bar of small curvature, in contact with the inner surface of a rigid cylinder. The bar forms an open circular ring, and the width of the gap produced by cutting the ring is insignificant. We assume that wear of the ring and cylinder surfaces takes place as the ring travels along the generatrix of the cylinder (during the working cycle of the piston) and the thickness of the ring is reduced as a result. We shall however neglect these changes while estimating the radial flexure $u(\theta, t)$ of the ring, and assume that the moment of inertia J remains approximately constant during the working period.

In the case of a simple flexure of circular bars of small curvature, the following equation holds /3/:

$$\frac{1}{r^2}\frac{\partial^2 u(\theta, t)}{\partial \theta^2} + \frac{u(\theta, t)}{r^2} = -\frac{M(\theta, t)}{EJ}$$

Here $M(\theta, t)$ is the bending moment of the bar and r is its radius of curvature. The moment is assumed positive when it causes compressive stresses in the outer fibers of the bar. We assume that the ends of the bar at the gap $(\theta = -\pi, \theta = \pi)$ are stress-free, i.e. the bending moment and tensile forces at these points are equal to zero. In this case the bending moment at any cross section of the ring will be the result of a continuous load $p(\theta, t)$ distributed along the outer surface of the ring and representing the pressure exerted on the ring by the cylinder. For the moment $M(\theta, t)$, we have

$$M(\theta, t) = -r^2 \int_{-\pi}^{\theta} p(\alpha, t)\sin(\theta - \alpha)\, d\alpha, \quad -\pi \leqslant \theta \leqslant \pi$$

Thus the pressure $p(\theta, t)$, radial bend of the ring $u(\theta, t)$ and its surface wear, are determined by the following system of equations:

$$\frac{\partial^2 u(\theta, t)}{\partial \theta^2} + u(\theta, t) = \frac{r^4}{EJ}\int_{-\pi}^{\theta} p(\alpha, t)\sin(\theta - \alpha)\, d\alpha \tag{1}$$

$$\frac{\partial u_*(\theta, t)}{\partial t} = kp(\theta, t), \quad u(\theta, t) = u(\theta, 0) - u_*(\theta, t) \tag{2}$$

The last equation represents the condition of contact between the ring and the cylinder. From (2) follows

$$\frac{\partial u(\theta, t)}{\partial t} = -kp(\theta, t) \tag{3}$$

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Let us differentiate (1) twice with respect to θ and add the result to the original equation (1). We obtain

$$\frac{\partial^4 u(\theta, t)}{\partial\theta^4} + 2\frac{\partial^2 u(\theta, t)}{\partial\theta^2} + u(\theta, t) = \frac{r^4}{EJ} p(\theta, t) \tag{4}$$

Equations (3) and (4) yield the following equation for determing the ring flexure:

$$\frac{\partial^4 u(\theta, t)}{\partial\theta^4} + 2\frac{\partial^2 u(\theta, t)}{\partial\theta^2} + u(\theta, t) + \frac{r^4}{EJk}\frac{\partial u(\theta, t)}{\partial t} = 0$$

We shall solve this equation by separating the variables, writing the unknown function $u(\theta, t)$ in the form

$$u(\theta, t) = U(\theta)\, T(t), \quad T(0) = 1 \tag{5}$$

Functions $T(t)$ and $U(\theta)$ are defined by the equations:

$$\frac{r^4}{EJk}\frac{dT}{dt} + \lambda^2 T = 0, \qquad \frac{d^4 U}{\partial\theta^4} + 2\frac{d^2 U}{d\theta^2} + U(1 - \lambda^2) = 0 \tag{6}$$

from which it follows that

$$T(t) = \exp\left(-\frac{\lambda^2 kEJ}{r^4} t\right), \quad U(\theta) = A \operatorname{sh} \Lambda^-\theta + B \operatorname{ch} \Lambda^-\theta + C \sin \Lambda^+\theta + D \cos \Lambda^+\theta, \quad \Lambda^\pm = \sqrt{\lambda \pm 1}, \ \ \lambda > 1 \tag{7}$$

Since the solution of (1) and (3) should be a function symmetrical in θ, we have

$$A = C = 0 \tag{8}$$

Let us find the values of the coefficients B and D, by satisfying to conditions of the problem. We use, as the first condition, one of the equations of equilibrium of the bar expressing the equality to zero of the projections of all external forces on the x-axis. The remaining equations of equilibrium of the ring are satisfied identically by virtue of the symmetry. We have

$$\int_{-\pi}^{\pi} p(\theta, t)\cos\theta\, d\theta = 0 \tag{9}$$

Using (3), (5), (7) and (8), we can write this condition in the form

$$B\int_{-\pi}^{\pi} \operatorname{ch} \Lambda^-\theta \cos\theta\, d\theta + D\int_{-\pi}^{\pi} \cos \Lambda^+\theta \cos\theta\, d\theta = 0$$

After computing the integrals, we have

$$B\Lambda^- \operatorname{sh} \Lambda^-\pi + D\Lambda^+ \sin \Lambda^+\pi = 0 \tag{10}$$

As the second condition, we use the condition that the moment is equal to zero at the ring ends at the gap

$$M(-\pi, t) = M(\pi, t) = 0 \tag{11}$$

Expressing the moment $M(\theta, t)$ using the displacements $u(\theta, t)$, we obtain

$$M(\theta, t) = -\frac{EJ}{r^2}\left[\frac{\partial^2 u(\theta, t)}{\partial\theta^2} + u(\theta, t)\right]$$

Condition (11) must hold at any instant of time. When $t = 0$, with due regard for the last relation we obtain

$$\left.\frac{d^2 U}{d\theta^2}\right|_{\theta=\pi} + U(\pi) = 0$$

or, taking into account the second relation of (7) and (8), we have

$$B \operatorname{ch} \Lambda^-\pi - D \cos \Lambda^+\pi = 0 \tag{12}$$

Equations (10) and (12) represent a system of two homogeneous equations for determining the coefficients B and D. The condition that the system has a nontrivial solution, yields the following characteristic equation for obtaining the eigenvalues:

$$\Lambda^+ \sin \Lambda^+\pi \operatorname{ch} \Lambda^-\pi + \Lambda^- \cos \Lambda^+\pi \operatorname{sh} \Lambda^-\pi = 0 \tag{13}$$

Let us denote the sequence of solutions of this equation by $\{\lambda_n\}, 1 < \lambda_2 < \ldots < \lambda_n \ldots$. From the solution of the system (10) and (12) we obtain the relations connecting B and D. The particular solutions assume, for $\lambda_n > 1$, the form

$$U_n(\theta) = \frac{\cos \Lambda_n^+\pi}{\operatorname{ch} \Lambda_n^-\pi} \operatorname{ch} \Lambda_n^-\theta + \cos \Lambda_n^+\theta, \quad (\Lambda_n^\pm = \sqrt{\lambda_n \pm 1}), \quad n = 2, 3, \ldots \tag{14}$$

It can be verified directly that $\lambda = 1$ is not an eigenvalue of the second equation of (6). In the case of $\lambda < 1$, the solution of the second equation of (6) can be written, taking into account the fact that the function $U(\theta)$ is even, in the form

$$U(\theta) = A \cos L^+ \theta + B \cos L^- \theta \quad (L^\pm = \sqrt{1 \pm \lambda}) \tag{15}$$

Let us find the values of the coefficients A and B. Fulfilling the conditions (9) and (11), we obtain a homogeneous system the characteristic equation of which has the form

$$-L^+ \sin L^+\pi \cos L^-\pi + L^- \sin L^-\pi \cos L^+\pi = 0$$

Clearly, $\lambda_0 = 0$ is a solution of this equation. The second root of this equation $\lambda_1 = 0.80$ with the accuracy of up to 0.005. The particular solutions corresponding to the eigenvalues λ_0, λ_1 have the form

$$U_0(\theta) = \cos\theta, \qquad U_1(\theta) = \cos L_1^+\theta + \frac{\cos L_1^+\pi}{\cos L_1^-\pi} \cos L_1^-\theta \quad (L_1^\pm = \sqrt{1 \pm \lambda_1}) \tag{16}$$

We shall prove that the particular solutions (14) are orthogonal. Performing the necessary manipulations we obtain, for any $\lambda_n \neq \lambda_m$

$$\int_{-\pi}^{\pi} U_m(\theta) U_n(\theta)\, d\theta = \cos\Lambda_n^+ \cos\Lambda_m^+ \frac{4}{\lambda_m^2 - \lambda_n^2} [F(\lambda_m) - F(\lambda_n)], \qquad (F(\lambda) = \lambda [\Lambda^- \operatorname{th} \Lambda^-\pi + \Lambda^+ \operatorname{tg} \Lambda^+\pi])$$

The right-hand side of this equation is equal to zero by virtue of the characteristic equation (13).

In the same manner we can show that the particular solutions (14) are also orthogonal with respect to the remaining two, mutually orthogonal particular solutions (16) of the second equation of (6).

Thus we see that all particular solutions of the second equation of (6) are orthogonal to each other. Expanding the known function defining the initial flexure $u(\theta, 0)$ into a series in the orthogonal system of functions (14) and (16), we obtain the coeffficients A_n

$$u(\theta, 0) = A_0 \cos\theta + \sum_{n=1}^{\infty} A_n U_n(\theta)$$

Next, taking into account (5) and (7), we obtain the expression for the radial flexure of the ring $u(\theta, t)$ at any instant of time

$$u(\theta, t) = A_0 \cos\theta + \sum_{n=1}^{\infty} A_n U_n(\theta) \exp\left(-\frac{\lambda_n^2 kEJ}{r^4} t\right) \tag{17}$$

Relations (3) and (17) yield the pressure at the ring perimeter at any instant of time

$$p(\theta, t) = \frac{EJ}{r^4} \sum_{n=1}^{\infty} A_n \lambda_n^2 U_n(\theta) \exp\left(-\frac{\lambda_n^2 kEJ}{r^4} t\right) \tag{18}$$

where the expressions for $U_n(\theta)$ $(n = 1, 2\ldots)$ are given by (14) and (16). The asymptotic expression for the pressure over long periods of time is easily obtained by restricting ourselves to one of several terms of the series in (18).

Analogous results are obtained in solving one-dimensional problems of a curved beam in contact with a half-plane, with the latter undergoing wear when the beam is displaced /4/.

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